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Resonant over-reflection of internal gravity waves from a thin shear layer

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In a previous paper (Grimshaw 1979) the resonant over-reflection of internal gravity waves from a vortex sheet was considered in the weakly nonlinear regime. It was shown there that the time evolution of the amplitude of the vortex sheet displacement was balanced by a cubic nonlinearity. For one vortex sheet mode, symmetrical with respect to the interface, it was shown that a steady finite-amplitude wave was possible. For the other, asymmetric modes, a singularity develops in a finite time. In the present paper, that analysis is extended by replacing the vortex sheet with a thin shear layer of thickness α^2, where α is the amplitude of the shear layer displacement. The effect of this extension is to introduce a linear growth rate term in the amplitude equation, which is otherwise unaltered. The linear growth rate can be computed from a formula due to Drazin & Howard (1966, p. 67). The effect on the modes is that the symmetric mode is linearly damped and requires sustained forcing to be observed, while the asymmetric modes are slightly destabilized by the linear term and, as in the vortex-sheet model, develop a singularity in finite time.

1. Introduction

The phenomena of over-reflection, in which a wave incident upon a shear layer generates a reflected wave of greater magnitude, has aroused considerable current interest. Acheson (1976) has reviewed the phenomena and, in particular, examined the energetics and described the way the excess reflected wave energy is extracted from the mean motion. A special case of over-reflection is resonant over-reflection in which, according to linear theory, there is no incident wave and the shear layer emits only outgoing waves. Lindzen (1974) drew attention to this phenomenon in a study of the stability of a vortex sheet in an infinite continuously stratified Boussinesq fluid of constant Brunt–Väisälä frequency. Lindzen commented on the possibility of a connection between over-reflection and instability of the basic shear profile. Subsequently many authors (e.g. Davis & Peltier 1976; Lalas, Einaudi & Fua 1976; Lalas & Einaudi 1976; Lindzen & Rosenthal 1976; Lindzen & Tung 1978; Lindzen, Farrell & Tung 1980) have demonstrated that the presence of one or more horizontal boundaries, or the presence of additional shear layers will imply the existence of slowly growing instabilities whose mechanism is successive wave over-reflection. In a recent survey of the stability properties of shear flows in unbounded media, Drazin, Zaturska & Banks (1979) have demonstrated that the resonant over-reflection mode of the vortex sheet is the limit of a slowly growing instability when the vortex sheet is replaced by a thin shear layer.

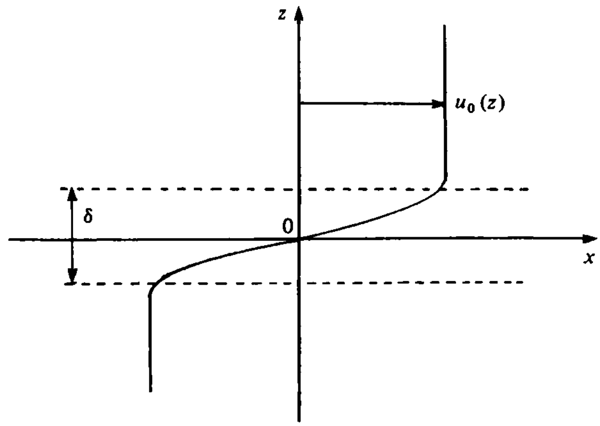

FIGURE 1. The co-ordinate system, and the profile of the basic velocity $u_0(z)$.

All the work referred to in the previous paragraph relates to linear theory. Grimshaw (1979) has examined the weakly nonlinear behaviour of the resonant over-reflection mode of the vortex sheet, and the purpose of this paper is to extend that analysis to the case when the vortex sheet is replaced by a thin shear layer. Davis & Peltier (1979) have examined numerically the nonlinear wave–wave interaction between the resonant over-reflection mode and the Kelvin–Helmholtz mode which occurs for higher wavenumbers. In this paper, however, we shall confine our attention to the resonant over-reflection mode.

The basic velocity profile $u_0(z)$ is sketched in figure 1. It demonstrates a transition from a velocity $+1$ as $z \to +\infty$ to -1 as $z \to -\infty$, with the transition being confined to a thin layer of thickness δ. The Brunt–Väisälä frequency $N(z)$ is a constant, taken to be 1, as $z \to \pm\infty$, but may vary within the shear layer. Throughout we shall use non-dimensional variables based on a velocity scale U_1, where $2U_1$ is the dimensional velocity discontinuity across the shear layer, a time scale N_1^{-1}, where N_1 is the dimensional Brunt–Väisälä frequency at infinity and a length scale $U_1 N_1^{-1}$. The equations of motion are those for an inviscid, and incompressible, Boussinesq fluid in the absence of rotation (cf. Grimshaw 1979). The Boussinesq parameter is $\sigma = U_1 N_1/g$, and is assumed to be small.

When the basic velocity profile is replaced by a vortex sheet ($\delta \to 0$), the linearized equations have solutions describing resonant over-reflection. The details are given by Grimshaw (1979), and we shall give only an outline here. If ζ is the vertical particle displacement, then in the regions outside the thin shear layer, denoted by the superscripts $\pm$ according as $z \gtrless 0$,

$$\zeta = \alpha A^{\pm} \exp\{ik(x-ct) \pm in^{\pm}z\} + \text{c.c.}, \tag{1.1}$$

where c.c. denotes the complex conjugate. Here α is a small parameter, measuring the amplitude of the waves, $k(>0)$ is the horizontal wavenumber, c is the horizontal phase speed and may be complex, and $n^{\pm}$ is the vertical wavenumber:

$$(n^{\pm})^2 = (W^{\pm})^{-2} - k^2, \quad W^{\pm} = c \mp 1. \tag{1.2}$$

The appropriate choice of sign for $n^{\pm}$ is determined by requiring that $\operatorname{Im} n^{\pm} > 0$ when $\operatorname{Im} c > 0$, and taking the limit $\operatorname{Im} c \to 0$. Thus

$$\operatorname{Im} n^{\pm} > 0, \quad \text{or} \quad \operatorname{Im} n^{\pm} = 0, \quad (\operatorname{Re} n^{\pm})\, W^{\pm} < 0. \tag{1.3}$$

For the linearized vortex sheet model, the boundary conditions at the vortex sheet imply that

$$\mathscr{D}(\omega, k) \equiv -in^{+}(\omega - k)^2 - in^{-}(\omega + k)^2 = 0, \tag{1.4}$$

where $\omega = kc$ is the frequency. The solutions of $\mathscr{D} = 0$ are

$$\text{(i)} \quad c = 0 \quad \text{for} \quad 0 < k^2 < 1, \tag{1.5a}$$

$$\text{(ii)} \quad c^2 = (2k^2)^{-1} - 1 \quad \text{for} \quad k^2 > \tfrac{1}{4}. \tag{1.5b}$$

The solution (i) describes resonant over-reflection, while the solution (ii) describes resonant over-reflexion for $\frac{1}{4} < k^2 < \frac{1}{2}$, and an unstable mode for $k^2 > \frac{1}{2}$.

The effect of replacing the vortex sheet by a thin shear layer of thickness δ is to perturb these vortex-sheet modes by $O(\delta)$. A procedure for calculating this perturbation has been described by Drazin & Howard (1966, p. 67) and results in an expression of the form

$$\mathscr{D}(\omega, k) + \delta k^2 L + O(\delta^2) = 0, \tag{1.6}$$

where L is an expression involving the structure of $u_0(z)$ and $N(z)$ within the thin shear layer (cf. (2.8*b*) which is the explicit expression for L). Equation (1.6) enables us to write $\omega = \omega_0 + \delta\omega_1$, where ω_0 (recall that $\omega = kc$) is the vortex-sheet mode (i) or (ii) and ω_1 is the perturbation; a simple calculation then determines ω_1 (cf. (2.10)). A derivation of (1.6) and the computation of ω_1 for some special cases is described in § 2. For typical shear-layer profiles we shall show that $\operatorname{Im}\omega_1 < 0$ for mode (i) and > 0 for mode (ii).

In the weakly nonlinear theory that follows in § 3 we shall allow only wavenumbers k such that $\frac{1}{4} < k^2 < \frac{1}{2}$, which corresponds to resonant over-reflection for both modes (other values of k are permitted for mode (i), see Grimshaw 1979). We let ϵ be a small parameter and put

$$T = \epsilon t, \quad Z = \epsilon z \tag{1.7}$$

and allow $A^{\pm}$ to depend on both T and Z. The aim of the analysis is to derive an amplitude equation for $A(T)$, where $A(T)$ is a measure of the displacement of the thin shear layer (approximately $A(T)$ is $A^{\pm}$ evaluated at $Z = 0$). The linear part of the amplitude equation may be deduced by replacing ω in (1.6) by $kc + i\epsilon\,\partial/\partial T$ and interpreting the result operationally. Hence we anticipate that the amplitude equation will be

$$\alpha\mathscr{D}(kc + i\epsilon\,\partial/\partial T, k)\,A + \alpha\delta k^2 LA = J + \alpha_0(\text{forcing}). \tag{1.8}$$

Here J represents the nonlinear term, and we shall show that J is proportional to $\alpha^3|A|^2A$, while the last term represents some weak forcing of amplitude α_0. Since the leading term on the left-hand side has magnitude $O(\epsilon\alpha)$, we see that the required balance between the time evolution, the shear-layer thickness, nonlinearity and forcing is $\epsilon = \delta = \alpha^2$ and $\alpha_0 = \alpha^3$. The amplitude equation is therefore

$$\partial A/\partial T = \gamma A + \beta|A|^2A + I. \tag{1.9}$$

Here $\gamma = -i\omega_1$, β is a coefficient whose value we shall derive in § 3, and I represents a forcing term.

For mode (i) it turns out that β is zero, while for (ii) the real part of β is positive. The corresponding solutions of the amplitude equation, and their implications are discussed in § 4. In appendix A the Drazin–Howard formula (1.6) is extended to

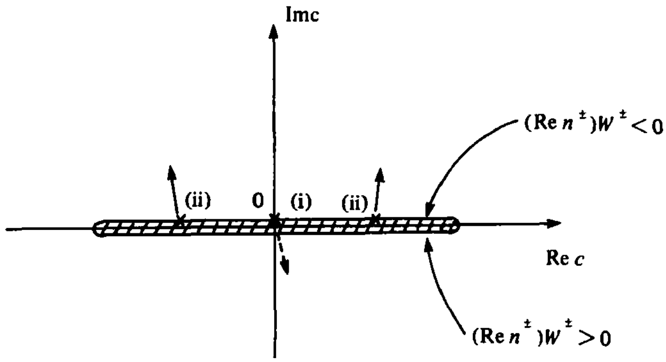

FIGURE 2. The complex c-plane, with two overlapping branch cuts along the real axis. The signs of Re $(n^{\pm})$ on the branch cuts is indicated in the diagram. × denotes the position of the resonant over-reflection modes (i) or (ii), for the vortex sheet model. The arrows denote the direction of the $O(\delta)$ thin shear-layer perturbation of these modes; ↑ denotes a perturbation on the Riemann surface Im $(n^{\pm}) > 0$; ↑ denotes a perturbation onto the Riemann surface Im $(n^{\pm}) < 0$.

include non-Boussinesq terms. In appendix B we present the full equations of motion in the generalized-Lagrangian-mean (GLM) formulation of Andrews & McIntyre (1978). A Lagrangian formulation of the equations is necessary as, within the thin shear layer, the thickness $\delta(O(\alpha^2))$ is smaller than the displacement of the shear layer. Throughout this paper, the co-ordinates x and z are Lagrangian co-ordinates, whose relation to the usual Eulerian co-ordinates is described in appendix B. For the linear theory of § 2 the distinction between the Lagrangian and Eulerian co-ordinates is of no consequence, but is essential to the nonlinear analysis of § 3.

2. Linear theory

The linear differential equation which governs the stability of the shear flow $u_0(z)$ in the presence of density stratification characterized by the Brunt–Väisälä frequency $N^2(z)$ is, in the Boussinesq approximation (Drazin & Howard 1966, p. 60),

$$\frac{\partial}{\partial z}\left\{W^2\frac{\partial\phi}{\partial z}\right\}+(N^2-k^2W^2)\,\phi=0, \tag{2.1a}$$

where

$$W=c-u_0. \tag{2.1b}$$

Here, if ζ is the vertical particle displacement, then ζ is proportional to

$$\mathrm{Re}\,\{\phi(z)\exp\,(ik(x-ct))\}.$$

As $z\to\pm\infty$, we shall suppose that $u_0\to U^{\pm}$ and $N\to N^{\pm}$; in § 1 and in subsequent sections $U^{\pm}=\pm 1$ and $N^{\pm}=1$, but in this section we shall allow a greater generality. We shall further suppose that u_0 and N approach these constant limits exponentially i.e. $|u_0-U^{+}|<\exp(-Mz)$ as $z\to\infty$ for some positive constant M, etc. Then, as $z\to\pm\infty$, it follows that

$$\phi\sim A^{\pm}\exp\,(\pm in^{\pm}z)\quad\text{as}\quad z\to\pm\infty, \tag{2.2a}$$

where

$$(n^{\pm})^2=(N^{\pm}/W^{\pm})^2-k^2,\quad W^{\pm}=c-U^{\pm}. \tag{2.2b, c}$$

The choice of sign for $n^{\pm}$ is determined by the criterion Im $(n^{\pm})>0$ when Im $c>0$ (1.3).

The complex c-plane must be cut along the real axis between the points where $n^{\pm}$ vanishes. In figure 2 we have sketched the uppermost Riemann surface on which $\mathrm{Im}(n^{\pm})$ are both positive; there are three other Riemann surfaces connected to the first across the branch cuts.

Let us now suppose that the shear layer is thin, and that u_0 and N are functions of $z^* = z/\delta$, where δ is a small parameter characterizing the thickness of the shear layer. The expressions (2.2a) are then outer expansions, valid for fixed z as $\delta \to 0$. They must be complemented by inner expansions, valid for fixed z^* as $\delta \to 0$. Matching these two expansions will yield the dispersion relation from which c as a function of k and δ can be determined. The dispersion relation we shall obtain by this procedure is identical with the Drazin–Howard formula for long waves in a shear flow (Drazin & Howard 1966, p. 67, or Drazin *et al.* 1979); their procedure is to continue the outer expansions into the shear-layer region by allowing $A^{\pm}$ to depend on z^*, and then to use the invariance of the Wronskian of the differential equation (2.1a) to determine the dispersion relation. We have preferred the matching procedure here as it extends naturally to the nonlinear calculation described in § 3.

In (2.1a), let ϕ be a function of z^*. Then (2.1a) becomes

$$\frac{\partial}{\partial z^*}\left\{W^2\frac{\partial\phi}{\partial z^*}\right\}+\delta^2(N^2-k^2W^2)\,\phi = 0. \tag{2.3}$$

It follows that

$$W^2\frac{\partial\phi}{\partial z^*} = \text{constant}+O(\delta^2). \tag{2.4}$$

Now $\partial\phi/\partial z^* = \delta\,\partial\phi/\partial z$ and matching with the outer expansions shows that $\partial\phi/\partial z^*$ is $O(\delta)$. Hence the constant of integration in (2.4) is $O(\delta)$, and (2.3) implies that the inner expansion is

$$\phi(z^*) = A+\delta B\int_0^{z^*}\frac{dz^*}{W^2}+\delta^2 A\int_0^{z^*}\frac{dz^*}{W^2}\int_0^{z^*}(k^2W^2-N^2)\,dz^*+O(\delta^3). \tag{2.5}$$

Here A and B are constants of integration, of the same order of magnitude as $A^{\pm}$.

Matching of the inner and outer expansions is now most readily accomplished by replacing z in (2.2a) with δz^*, expanding the result in powers of δ, and equating these expansions with (2.5). Strictly speaking, the method of intermediate expansions (Cole 1968, p. 10) should be used, but the simpler method described above leads directly to the Drazin–Howard formula. We find that

$$\begin{aligned}
&A^{\pm}\{1\pm\delta i n^{\pm}z^*-\tfrac{1}{2}\delta^2(n^{\pm})^2z^{*2}+O(\delta^3)\}\\
&\sim A+\frac{\delta B}{(W^{\pm})^2}z^*+\delta B\int_0^{\pm\infty}\left(\frac{1}{W^2}-\frac{1}{W^{\pm 2}}\right)dz^*\\
&\quad-\tfrac{1}{2}\delta^2A(n^{\pm})^2z^{*2}+\frac{\delta^2Az^*}{W^{\pm 2}}\int_0^{\pm\infty}(k^2W^2-N^2-k^2W^{\pm 2}-N^{\pm 2})\,dz^*\\
&\quad+\delta^2[\ldots]+O(\delta^3).
\end{aligned} \tag{2.6}$$

The term $\delta^2[\ldots]$ is a constant, whose explicit value is omitted as it will not be needed subsequently. Matching the terms proportional to z^*, and the constant terms, yields, respectively,

$$B+\delta A\int_0^{\pm\infty}(k^2W^2-N^2-k^2W^{\pm 2}-N^{\pm 2})\,dz^* = \pm i n^{\pm}W^{\pm 2}A^{\pm}, \tag{2.7a}$$

$$A+\delta B\int_0^{\pm\infty}\left(\frac{1}{W^2}-\frac{1}{W^{\pm 2}}\right)dz^* = A^{\pm}. \tag{2.7b}$$

We have omitted terms of $O(\delta^2)$. Note that (2.7 *b*) implies that the terms proportional to $\delta^2 z^{*2}$ in (2.6) are already matched. Equations (2.7 *a*, *b*) are four equations for the four constants A, B and $A^{\pm}$. Eliminating these constants gives the dispersion relation

$$\mathscr{D}(\omega, k) + \delta k^2 L + O(\delta^2) = 0, \tag{2.8a}$$

where
$$\begin{aligned} L = {} & k^2 \int_0^{\infty} (W^2 - W^{+2})\, dz^* + k^2 \int_{-\infty}^{0} (W^2 - W^{-2})\, dz^* \\ & - \int_0^{\infty} (N^2 - N^{+2})\, dz^* - \int_{-\infty}^{0} (N^2 - N^{-2})\, dz^* \\ & + n^+ n^- W^{-2} \int_0^{\infty} \left(1 - \frac{W^{+2}}{W^2}\right) dz^* + n^+ n^- W^{+2} \int_{-\infty}^{0} \left(1 - \frac{W^{-2}}{W^2}\right) dz^*, \end{aligned} \tag{2.8b}$$

and
$$\mathscr{D}(\omega, k) = -in^+ k^2 W^{+2} - in^- k^2 W^{-2}. \tag{2.8c}$$

Equation (2.8*a*) is the Drazin–Howard formula for long waves in a shear flow, and may readily be used to calculate the $O(\delta)$ perturbation of the vortex sheet models given by $\mathscr{D} = 0$. Thus let $\omega_0(k)$ be a vortex sheet mode, and put

$$\omega = \omega_0(k) + \delta \omega_1(k) + O(\delta^2). \tag{2.9}$$

Then (2.8*a*) shows that
$$\omega_1 \mathscr{D}_{\omega}(\omega_0, k) = -k^2 L_0, \tag{2.10}$$

where L_0 is L evaluated at $\omega = \omega_0$. For the case $U^{\pm} = \pm 1$ and $N^{\pm} = 1$ when $\mathscr{D}$ is given by (1.4), we note that

$$\mathscr{D}_{\omega}(\omega_0, k) = \begin{cases} 2ik(1-2k^2)(1-k^2)^{-\frac{1}{2}} & \text{for mode (i)}, \quad (2.11a) \\ -8ik^2(1-2k^2)(4k^2-1)^{-1} & \text{for mode (ii)}. \quad (2.11b) \end{cases}$$

For this same case, when $k^2 \approx \frac{1}{2}$, $\mathscr{D}_{\omega}(\omega_0, k)$ is zero. There is a triple coalescence of modes, i.e. mode (i) coalesces with both modes (ii), and for all three modes $\omega \approx 0$. Equation (2.10) is replaced by

$$-8i\omega\left(\omega^2 + \frac{(k-k_c)}{k_c}\right) \approx -\delta k^2 L_0, \quad k_c^2 = \tfrac{1}{2}. \tag{2.12}$$

Also, for $k^2 \approx \frac{1}{4}$, $\mathscr{D}_{\omega}(\omega_0, k)$ is infinite for mode (ii) and a separate analysis is needed near this wavenumber.

Further progress depends on the evaluation of L_0, which in turn depends on specifying $u_0(z^*)$ and $N(z^*)$. Assuming that $U^{\pm} = \pm 1$ and $N^{\pm} = 1$, we shall consider the three cases:

$$(a) \quad u_0(z^*) = \begin{cases} z^* & \text{for } |z^*| < 1, \quad N(z^*) = 1, \\ \pm 1 & \text{for } z^* \gtrless \pm 1; \end{cases} \tag{2.13a}$$

$$\left.\begin{aligned} (b) \quad & u_0(z^*) \quad \text{as in } (a) \text{ above}, \\ & N(z^*) = 0 \quad \text{for} \quad |z^*| \leqslant 1, \quad N(z^*) = 1 \quad \text{for} \quad |z^*| > 1; \end{aligned}\right\} \tag{2.13b}$$

$$(c) \quad u_0(z^*) = \tanh z^*, \quad N(z^*) = 1. \tag{2.13c}$$

Note that, although the formula (2.8*b*) has been derived under the condition that $u_0(z^*)$ and $N(z^*)$ are smooth functions of z^*, the formula is readily extended to the case when $N(z^*)$ or $\partial u_0(z^*)/\partial z^*$ have a finite number of simple jump discontinuities.

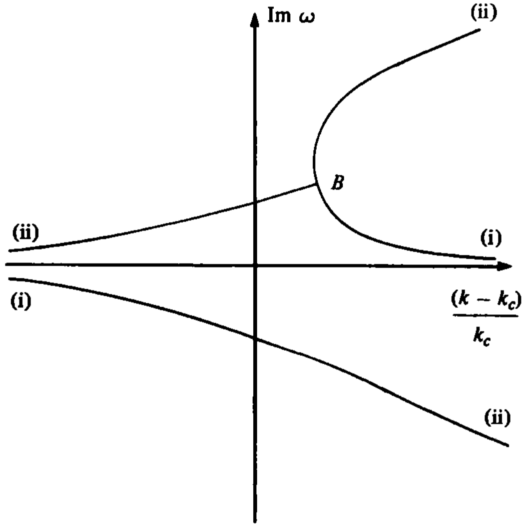

FIGURE 3. The graph of $\operatorname{Im}\omega$ when ω is given by the cubic equation (2.15). For $k-k_c < 0$, and up to the bifurcation point B, the two branches of mode (ii) have $\operatorname{Re}\omega \neq 0$; beyond the bifurcation point B, and for the whole branch in $\operatorname{Im}\omega < 0$, $\operatorname{Re}\omega = 0$.

Cases (a) and (c) have been discussed by Drazin *et al.* (1979). We find that

$$(a)\quad \left.\begin{array}{ll}\text{mode (i)}, & L_0 = \tfrac{8}{3}k^2-4,\\ \text{mode (ii)}, & L_0 = -\tfrac{16}{3}k^2;\end{array}\right\} \tag{2.14a}$$

$$(b)\quad \left.\begin{array}{ll}\text{mode (i)}, & L_0 = \tfrac{8}{3}k^2-2,\\ \text{mode (ii)}, & L_0 = -\tfrac{16}{3}k^2+2;\end{array}\right\} \tag{2.14b}$$

$$(c)\quad \left.\begin{array}{ll}\text{mode (i)}, & L_0 = -2,\\ \text{mode (ii)}, & L_0 = -2k^2\left(2+c\ln\left(\dfrac{1+c}{1-c}\right)-i\pi c\right).\end{array}\right\} \tag{2.14c}$$

Recalling that $\mathscr{D}_\omega$ is given by (2.11a, b), we see that in all three cases $\operatorname{Im}\omega_1 < 0$ for mode (i) when $k^2 < \frac{1}{2}$ (for $\frac{1}{2} < k^2 < 1$ and for cases (a) and (c) $\operatorname{Im}\omega_1 > 0$ while for case (b) $\operatorname{Im}\omega_1 > 0$ for $\frac{1}{2} < k^2 < \frac{3}{4}$, and $\operatorname{Im}\omega_1 < 0$ for $\frac{3}{4} < k^2 < 1$), and in all three cases $\operatorname{Im}\omega_1 > 0$ for mode (ii). Thus in all three cases for $\frac{1}{4} < k^2 < \frac{1}{2}$, mode (i) is stabilized by the $O(\delta)$ perturbation, while mode (ii) is destabilized. The situation is described schematically in figure 2.

For $k^2 \approx \frac{1}{2}$, $L_0 = -\frac{8}{3}$ for case (a), $-\frac{2}{3}$ for case (b) and $-\frac{16}{5}$ for case (c). In all three cases L_0 is real and negative and (2.12) becomes

$$\omega\left(\omega^2+\frac{(k-k_c)}{k_c}\right) \approx \frac{i\delta k^2|L_0|}{8}, \quad k_c^2 = \tfrac{1}{2}. \tag{2.15}$$

The solutions of this cubic equation for ω are sketched schematically in figure 3. The graphs show that, as k increases through the critical value k_c from the stable region $k < k_c$ to the unstable region $k > k_c$, mode (i) is transformed into the stable branch of mode (ii), while the two branches of mode (ii) for $k < k_c$ coalesce near $k = k_c$, and then bifurcate, with one branch becoming the unstable branch of mode (ii), and the other becoming mode (i).

3. Weakly nonlinear theory

For the weakly nonlinear theory the amplitude $A^{\pm}$ of the vertical particle displacement ζ in equation (1.1) is a function of the slow time and space variables, T and Z respectively (1.7). All field variables are expressed as Fourier series in the phase variable $k(x-ct)$, with coefficients which, for the outer expansions (valid in the region outside the thin shear layer), are functions of T, z and Z. Thus, for example,

$$\zeta = \sum_{m=1}^{\infty} \zeta_m(T, z, Z) \exp\{imk(x-ct)\} + \text{c.c.}, \tag{3.1}$$

where c.c. denotes the complex conjugate. The coefficients in these equations are determined by substituting these Fourier series into the equations of motion and equating like Fourier components. For the Eulerian equations of motion, the procedure and results are described by Grimshaw (1979). Here, for reasons given below, we have chosen to use a Lagrangian formulation of the equations of motion. For the outer expansions, this leads to some differences in details from the Eulerian formulation, but no major differences in the results. Hence we shall give only a brief outline.

The Lagrangian formulation we shall use is the generalized-Lagrangian-mean formulation of Andrews & McIntyre (1978), which is described in appendix B. The field variables are the Lagrangian mean quantities, $\bar{u}^{\mathrm{L}}$ (mean horizontal velocity), $\bar{w}^{\mathrm{L}}$ (mean vertical velocity), $\bar{p}^{\mathrm{L}}$ (mean pressure) and $\bar{\rho}^{\mathrm{L}}$ (mean density); here the leading terms in $\bar{u}^{\mathrm{L}}$, $\bar{p}^{\mathrm{L}}$ and $\bar{\rho}^{\mathrm{L}}$ are $u_0(z)$ (the basic shear flow), $p_0(z)$ (the basic pressure profile) and $\rho_0(z)$ (the basic density profile). The equations for these mean quantities are (B 5*a*), (B 7) and (B 10*a*, *b*). Relative to this mean flow there are the field variables which describe the wave motion; these variables are ξ (horizontal particle displacement), ζ (vertical partical displacement) and q (a pressure perturbation). The equations for these variables are (B 6*b*) and (B 11*a*, *b*). It is now convenient to write these equations for the outer expansions as $z \to \pm\infty$ in the following form

$$\frac{\partial \xi}{\partial x} + \frac{\partial \zeta}{\partial z} + G = 0, \tag{3.2a}$$

$$\left(\frac{\partial}{\partial t} \pm \frac{\partial}{\partial x}\right)^2 \xi + \frac{\partial q}{\partial x} + F_H = 0, \tag{3.2b}$$

$$\left(\frac{\partial}{\partial t} \pm \frac{\partial}{\partial x}\right)^2 \zeta + \frac{\partial q}{\partial z} + \zeta + F_V = 0. \tag{3.2c}$$

Here G, F_H, and F_V are nonlinear expressions which we shall not display explicitly; they may be readily deduced from the exact equations (B 6*b*) and (B 11*a*, *b*). We have also used the limits $u_0(z) \to \pm 1$ and $N \to 1$ as $z \to \pm\infty$, and used the Boussinesq approximation. Elimination of ξ and q gives the single equation

$$L^{\pm}\left\{\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right\} \zeta + M^{\pm} = 0, \tag{3.3a}$$

where
$$L^{\pm}\left\{\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right\} = -\left\{\frac{\partial}{\partial t} \pm \frac{\partial}{\partial x}\right\}^2 \left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right\} - \frac{\partial^2}{\partial x^2}, \tag{3.3b}$$

and
$$M^{\pm} = -\left\{\frac{\partial}{\partial t} \pm \frac{\partial}{\partial x}\right\}^2 \frac{\partial G}{\partial z} - \frac{\partial^2 F_V}{\partial x^2} + \frac{\partial^2 F_H}{\partial x\, \partial z}. \tag{3.3c}$$

Here $L^{\pm}$ is a linear operator, and $M^{\pm}$ contains the nonlinear terms. Substituting the Fourier series (3.1) into (3.3*a*), and equating like Fourier components, it follows that

$$L^{\pm}\left\{-imkc+\epsilon\frac{\partial}{\partial T},\ imk,\ \frac{\partial}{\partial z}+\epsilon\frac{\partial}{\partial Z}\right\}\zeta_m = M^{\pm}_m, \tag{3.4}$$

where $M^{\pm}_m$ is the mth Fourier component of $M^{\pm}$.

For the Fourier component $m = 1$, it may be shown that $M^{\pm}_1$ is $O(\alpha^3)$ and so

$$\zeta_1 = \alpha A^{\pm}(T,Z)\exp(\pm in^{\pm}z)+\alpha_0 I^{\pm}(T,Z)\exp(\mp in^{\pm}z)+\alpha^3\zeta_1^{(3)}+O(\alpha^5), \tag{3.5}$$

where $n^{\pm}$ is defined by (1.2), and the terms $I^{\pm}$ represent forcing by incident waves; $\zeta_1^{(3)}$ is a term forced by $M^{\pm}_1$ at $O(\alpha^3)$. For $m = 2$ it may be shown that $M^{\pm}_2$ is $O(\alpha^5)$, and so

$$\zeta_2 = \alpha^2 A^{\pm}_2(T,Z)\exp(\pm in^{\pm}_2 z)+O(\alpha^4), \tag{3.6a}$$

where
$$(n^{\pm}_2)^2 = (c\mp 1)^{-2}-4k^2. \tag{3.6b}$$

Here $A^{\pm}_2$ will be determined by matching with the corresponding Fourier component in the shear layer, and the factor α^2 has been inserted in anticipation of this matching. A discussion of the value of $n^{\pm}_2$ for each mode (i) or (ii), and the role this Fourier component plays in the amplitude equation is given by Grimshaw (1979).

The mean flow, or Fourier component $m = 0$, is determined from the mean flow equations (B 5*a*), (B 7) and (B 10*a*, *b*). It may be shown from (B 5*a*) and (B 6*a*) that $\overline{w}^{\mathrm{L}}$ is $O(\epsilon\alpha^4)$, and then it follows from (B 7) that $\overline{\rho}^{\mathrm{L}} = \rho_0(z)+O(\alpha^4)$. It is readily seen that the solution of (B 10*a*) is $\overline{u}^{\mathrm{L}} = u_0(z)+\mathscr{P}_H$, as the mean flow is independent of x. Evaluating $\mathscr{P}_H$, it follows that

$$\overline{u}^{\mathrm{L}} = \pm 1+\frac{2\alpha^2|A^{\pm}|^2}{W^{\pm}}+O(\alpha^4), \tag{3.7}$$

since $u_0(z)\to\pm 1$ as $z\to\pm\infty$. Finally it may be shown from (B 10*b*) that

$$\overline{p}^{\mathrm{L}} = p_0(z)+2\alpha^2(n^{\pm}W^{\pm})^2|A^{\pm}|^2+O(\alpha^4). \tag{3.8}$$

Returning to the Fourier component $m = 1$, we find that

$$M^{\pm}_1 = \frac{4\alpha^2k^2|A^{\pm}|^2\eta_1}{(W^{\pm})^2}+\tfrac{3}{2}in^{\pm}_2 k^2W^{\pm 2}(2n^{\pm}-n^{\pm}_2)^2\zeta_2\overline{\zeta}_1+O(\alpha^5). \tag{3.9}$$

The first term, being proportional to ζ_1, determines the variation of $A^{\pm}$ with T and Z,

$$\epsilon\frac{\partial A^{\pm}}{\partial Z} = \frac{\pm\epsilon}{kn^{\pm}W^{\pm 3}}\frac{\partial A^{\pm}}{\partial T}\pm\frac{2i\alpha^2|A^{\pm}|^2A^{\pm}}{n^{\pm}W^{\pm 4}}. \tag{3.10}$$

This equation is identical with the corresponding equation in the Eulerian formulation (Grimshaw 1979), where it was shown that modulations in $|A^{\pm}|$ propagate vertically with the local group velocity, while phase changes in $A^{\pm}$ are determined by an amplitude-dependent Doppler shift in the local frequency due to the wave-induced component of $\overline{u}^{\mathrm{L}}$ (3.7). The second term in $M^{\pm}_1$ (3.9) is responsible for the term $\zeta_1^{(3)}$ in (3.5), and we find that

$$\zeta_1^{(3)} = -\tfrac{3}{2}i(2n^{\pm}-n^{\pm}_2)\zeta_2\overline{\zeta}_1. \tag{3.11}$$

For the inner expansion within the thin shear layer we put $z^* = z/\delta$. Since the thickness of the shear layer is $O(\delta)$, while the amplitude of the motion of the shear layer is $O(\alpha)$, and our scaling anticipates that $\delta = \alpha^2$, we see that the thickness of the shear layer is an order of magnitude smaller than its displacement. It is for this reason that an Eulerian formulation poses difficulties, and we have preferred to use a Lagrangian formulation. With $z^* = z/\delta$, the equations (B 6*b*) and (B 11*a*, *b*) become, in the Boussinesq approximation,

$$\frac{\partial \zeta}{\partial z^*} + \delta \frac{\partial \xi}{\partial x} + \left[\frac{\partial \xi}{\partial x}\frac{\partial \zeta}{\partial z^*} - \frac{\partial \zeta}{\partial x}\frac{\partial \xi}{\partial z^*}\right] = 0, \tag{3.12a}$$

$$\left\{\epsilon \frac{\partial}{\partial T} - W\frac{\partial}{\partial x}\right\}^2 \xi + \frac{\partial q}{\partial x} + \left[W^2 \frac{\partial^2 \xi}{\partial x^2}\frac{\partial \xi}{\partial x} + W^2 \frac{\partial^2 \zeta}{\partial x^2}\frac{\partial \zeta}{\partial x}\right] + 2(\bar{u}^{\mathrm{L}} - u_0)\, W \frac{\partial^2 \xi}{\partial x^2} + \ldots = 0, \tag{3.12b}$$

$$\delta\left(\epsilon \frac{\partial}{\partial T} - W\frac{\partial}{\partial x}\right)^2 \zeta + \frac{\partial q}{\partial z^*} + \delta N^2 \zeta + \left[W^2 \frac{\partial^2 \xi}{\partial x^2}\frac{\partial \xi}{\partial z^*} + W^2 \frac{\partial^2 \zeta}{\partial x^2}\frac{\partial \zeta}{\partial z^*}\right] + \ldots = 0, \tag{3.12c}$$

where the omitted terms are higher-order nonlinear terms not needed in the subsequent analysis, and the square-bracket notation indicates that the Fourier component $m = 0$ is to be deleted from the contained expression. For these equations we again seek a Fourier series expansion of the form

$$\zeta = \sum_{m=1}^{\infty} \zeta_1(T, z^*) \exp\{imk(x - ct)\} + \text{c.c.} \tag{3.13}$$

It follows immediately from (3.12*a*, *b*, *c*) that

$$\zeta_1 = \alpha A(T) + O(\alpha^3), \quad \xi_1 = \frac{i\alpha B(T)}{kW^2} + O(\alpha^3), \quad q_1 = \alpha B(T) + O(\alpha^3); \tag{3.14a, b, c}$$

$A(T)$ is the amplitude of the displacement of the vortex sheet, to which the shear layer reduces in the limit $\delta \to 0$. Matching with the outer expansions, e.g. (3.5), shows that

$$A^{\pm}(T, 0) = A(T) + O(\alpha^2), \tag{3.15a}$$

$$\pm in^{\pm} W^{\pm 2} A^{\pm}(T, 0) = B(T) + O(\alpha^2). \tag{3.15b}$$

The dispersion relation (1.4) for the vortex-sheet limit is readily deduced from (3.15*a*, *b*).

For $m = 2$, it may be shown from (3.12*a*, *b*, *c*) that

$$\zeta_2 = \alpha^2 A_2(T) - \frac{\alpha^2 AB}{W^2} + O(\alpha^4), \tag{3.16a}$$

$$2ik\xi_2 = -\frac{\alpha^2 B_2(T)}{W^2} + \frac{\alpha^2}{2}\left(\frac{B^2}{W^4} + k^2 A^2\right) + O(\alpha^4), \tag{3.16b}$$

$$q_2 = \alpha^2 B_2(T) - \frac{\alpha^2 B^2}{W^2} + O(\alpha^4). \tag{3.16c}$$

Matching with the outer expansions, e.g. (3.6*a*), shows that

$$A_2^{\pm}(T, 0) = A_2(T) \mp in^{\pm} A^2 + O(\alpha^2), \tag{3.17a}$$

$$\pm in_2^{\pm} W^{\pm 2} A_2^{\pm}(T, 0) = B_2(T) + \{\tfrac{1}{2} - k^2 W^{\pm 2}\} A^2 + O(\alpha^2). \tag{3.17b}$$

There are four equations for $A_2^{\pm}$, A_2 and B_2. Solving these, we find that, for each mode,

$$\text{(i)}\quad A_2 = i(1-k^2)^{\frac{1}{2}} A^2 + O(\alpha^2), \quad A_2^{\pm} = O(\alpha^2), \tag{3.18a}$$

$$\text{(ii)}\quad A_2 = \frac{iA^2}{\Delta}\{-4k^2c + k(1-c^2)(n_2^+ + n_2^-)\} + O(\alpha^2), \tag{3.18b}$$

$$A_2^{\pm}(T,0) = \frac{iA^2}{\Delta}\left\{-4k^2c \mp 4kcn_2^{\mp}\left(\frac{1\pm c}{1\mp c}\right)\right\} + O(\alpha^2), \tag{3.18c}$$

where

$$\Delta = n_2^+(1-c)^2 + n_2^-(1+c)^2. \tag{3.18d}$$

These results agree with the Eulerian calculations of Grimshaw (1979).

The mean flow, or Fourier component $m = 0$ is determined from the mean flow equations (B 5a), (B 7) and (B 10a, b), with the substitution $z^* = z/\delta$. It may be shown from (B 5a) and (B 6a) that $\overline{w}^{\mathrm{L}}$ is $O(\epsilon\alpha^4)$, and matches with the outer expansions, although we shall not display the details of this calculation. Then it follows from (B 7) that $\overline{\rho}^{\mathrm{L}} = \rho_0(z) + O(\alpha^4)$, and also matches with the outer expansions. The solution of (B 10a) is $\overline{u}^{\mathrm{L}} = u_0(z^*) + \mathscr{P}_H$, and this clearly matches with the outer expansions, as it is the same solution. Within the shear layer

$$\overline{u}^{\mathrm{L}} = u_0(z^*) + 2\alpha^2 W\left\{k^2|A|^2 + \frac{|B|^2}{W^4}\right\} + O(\alpha^4). \tag{3.19}$$

Finally it may be shown from (B 10b) that

$$\overline{p}^{\mathrm{L}} = p_0(z^*) + 2\alpha^2\frac{|B|^2}{W^2} + O(\alpha^4), \tag{3.20}$$

which matches with the outer expansions (3.8), on using (3.15b).

Returning to the Fourier component $m = 1$, we find from (3.12a) that

$$\zeta_1 = \alpha A + \alpha\delta B\int_0^{z^*}\frac{dz^*}{W^2} + \alpha^3 A_3 + O(\alpha^5), \tag{3.21a}$$

where

$$A_3 = \left\{\frac{2A_2\bar{B}}{W^2} + \frac{B_2\bar{A}}{2W^2} - \frac{3|B|^2A}{2W^4} - \frac{B^2\bar{A}}{4W^4} - \frac{k^2|A|^2A}{4}\right\}. \tag{3.21b}$$

Similarly we find from (2.12c) that

$$q_1 = \alpha B - \alpha\delta A\int_0^{z^*}(N^2 - k^2W^2)\,dz^* + \alpha^2 B_3 + O(\alpha^5), \tag{3.22a}$$

where

$$B_3 = \left\{\frac{5}{2}\frac{B_2\bar{B}}{W^2} - \frac{3}{4}\frac{|B|^2B}{W^4}\right\}; \tag{3.22b}$$

ξ_1 may then be found from (3.12b). These expressions, (3.21) for ζ_1 and (3.22) for q_1, are then matched to the outer expansions, (3.5) for ζ_1 and a similar expression for q_1 deduced from (3.2b). This matching process produces a set of equations for the variables A, B and $A^{\pm}(Z = 0)$, which, for the terms proportional to δ are identical with equations (2.7a, b), but now contain some additional terms of $O(\alpha^2)$ (the nonlinear terms), $O(\epsilon)$ (terms proportional to $\partial A/\partial T$), and terms involving the incident wave packets. Eliminating B and $A^{\pm}(Z = 0)$, we find that

$$\alpha\mathscr{D}\left(kc + i\epsilon\frac{\partial}{\partial T}, k\right)A = \alpha_0(\mathscr{D}^+I^+ + \mathscr{D}^-I^-\} - \alpha\delta k^2 L + J, \tag{3.23}$$

where $\mathscr{D}$ is defined by (2.8*c*), L is defined by (2.8*b*),

$$\mathscr{D}^{\pm} = 2in^{\pm}kW^{\pm 2} \tag{3.24}$$

and J is the nonlinear expression

$$\begin{aligned} J = {} & ik[F_{H1}]_{-}^{+} + k^2[W^2G_1]_{-}^{+} - i\alpha^3k^2[nW^2\zeta_1^{(3)}]_{-}^{+} \\ & + i\alpha^3\left[k^2W^2\frac{\partial\zeta_1^{(3)}}{\partial z}\right]_{-}^{+} + i\alpha^3k^2[nW^2A_3]_{-}^{+} \\ & - \alpha^3k^2[B_3]_{-}^{+} + O(\alpha^5). \end{aligned} \tag{3.25}$$

Here $[\ldots]_{-}^{+}$ denotes the discontinuity in the contained expression across the thin shear layer. Evaluating J, we find that

$$J = \frac{8ick^4\alpha^3|A|^2A}{\Delta}\left\{8kc - \frac{3(1+c^2)\Delta}{(1-c^2)} - \frac{4c}{(1-c^2)}(n_2^+(1-c)^2 - n_2^-(1+c)^2)\right\} + O(\alpha^5), \tag{3.26}$$

which agrees with the Eulerian calculation of Grimshaw (1979).

To conclude this section we note that the expansion in the amplitude parameter α fails in the vicinity of the critical level where $W \approx 0$. Since the formulae (2.14*a, b, c*) show that $\operatorname{Im} c \neq 0$, our nonlinear expressions remain determinate, but expressions such as (3.16*a, b, c*), (3.21*b*) and (3.22*b*) will fail when W is $O(\alpha^{\frac{1}{2}})$. A separate critical-layer analysis is needed, the thickness of this layer being $O(\alpha^{\frac{1}{2}})$ in the z^* co-ordinate, or $O(\alpha^{\frac{3}{2}})$ in the z co-ordinate. We shall not attempt this analysis here, as our concern is to evaluate J, and, since none of the nonlinear expressions to $O(\alpha^3)$ involve integrals across the critical layer, we have been able to evaluate J without recourse to a critical-layer analysis. In contrast the Drazin–Howard formula L (2.8*b*) for the perturbation of the vortex-sheet modes does involve integrals across the critical layer, and to evaluate these integrals we have invoked the fact that $\operatorname{Im} c$, although small, is not zero.

4. Discussion of the amplitude equation

The amplitude equation is (1.9), or (4.1) below. The coefficient β is found by deriving the amplitude equation in the form (1.8) or (3.23), where $\mathscr{D}_\omega$ is given by (2.11*a, b*) and $\mathscr{D}^{\pm}$ by (3.24), and then combining this with J as given by (3.26). The result is

$$\frac{\partial A}{\partial T} = \gamma A + \beta|A|^2A + I, \tag{4.1}$$

where the coefficient β is given by

$$\beta = \begin{cases} 0 \quad \text{for mode (i),} & (4.2a) \\ -\dfrac{3ik}{2c} + 4ik^3\left\{\dfrac{2k(1-c^2) - n_2^+(1-c)^2 + n_2^-(1+c)^2}{n_2^+(1-c)^2 + n_2^-(1+c)^2}\right\} \quad \text{for mode (ii).} & (4.2b) \end{cases}$$

The coefficient γ is $-i\omega_1$, where ω_1 is given by (2.10), and I is a linear combination of I^+ and I^- with coefficients $\mathscr{D}^{\pm}/\mathscr{D}_\omega$ (see (2.11*a, b*) and (3.24)). The coefficient β is identical with the coefficient derived by Grimshaw (1979) for the nonlinear term of the vortex-sheet model, and so the only effect on the amplitude equation of replacing the vortex-sheet model with the thin shear layer is the introduction of the linear term γA. In § 2 it was shown for three typical shear layer profiles that $\operatorname{Re}\gamma$ is negative for mode (i), and positive for mode (ii). In the subsequent discussion we shall assume these signs for $\mathrm{Re}\,\gamma$.

For mode (i), when $\beta = 0$ and $\mathrm{Re}\,\gamma < 0$ the solution is

$$A = A_0 \exp(\gamma T) + \int_0^T I(T') \exp(\gamma(T - T'))\, dT', \tag{4.3}$$

where A_0 is the amplitude at $T = 0$. If the forcing has finite extent ($I = 0$ for $T > T_0$ say), then ultimately $A \to 0$ as $T \to \infty$. However, if the forcing is maintained and $I \to$ constant as $T \to \infty$, then $A \to -I/\gamma$ as $T \to \infty$. Thus, for mode (i), sustained forcing will lead to a steady over-reflection mode.

For mode (ii), it was shown by Grimshaw (1979) that $\beta_R = \mathrm{Re}(\beta)$ is positive. If there is no forcing, or if the time scale of the incident wave packets is very short so that I may be approximated by $A_0 \delta(T)$, where $\delta(T)$ is the delta function, then the solution of (3.1) is

$$A = A_0 \exp(\gamma T)\left\{1 + \frac{\beta_R |A_0|^2}{\gamma_R}(1 - \exp(2\gamma_R T))\right\}^{-\nu} \quad \text{for} \quad T \geqslant 0, \tag{4.4a}$$

where

$$\nu = \frac{\beta}{2\beta_R} \quad \text{and} \quad \gamma_R = \mathrm{Re}(\gamma). \tag{4.4b}$$

Here γ_R is positive, and thus the solution develops a singularity in a time T_∞, where

$$T_\infty = \frac{1}{2\gamma_R} \ln\left(1 + \frac{\gamma_R}{\beta_R |A_0|^2}\right). \tag{4.5}$$

As $T \to T_\infty$ both $|A|$ and $\arg A$ approach infinity as A spirals outwards in the complex-A plane. The solution is qualitatively similar to the case $\gamma = 0$ considered by Grimshaw (1979), where there is a discussion of the application of this result to observed waves in the atmosphere. Here we note that, as $\gamma_R \to 0$, $T_\infty \to 1/2\beta_R |A_0|^2$ and, as γ_R increases from zero, T_∞ decreases.

If I is not zero, then in general (4.1) must be integrated numerically. If the time scale of the incident wave packets is very long so that I may be approximated by a constant, then there is an equilibrium solution of (4.1). However, this equilibrium solution is unstable, and all solutions develop a singularity in finite time. Close to this singularity the solution will be described by an expression similar to (4.4a). When $\gamma = 0$, a detailed analysis of this case is described by Grimshaw (1979); for $\gamma \neq 0$ the analysis is similar but will not be repeated here.

In summary, the effect of replacing the vortex sheet model with the thin shear layer, is to introduce a linear growth rate term (γA) in the amplitude equation, which is otherwise unaltered. Further, we find that for mode (i) $\mathrm{Re}\,\gamma < 0$ for typical shear layers, and β is zero; thus this mode requires sustained, although weak, forcing to be observed. In contrast, for mode (ii) $\mathrm{Re}\,\gamma > 0$ and $\mathrm{Re}\,\beta > 0$; this mode develops a singularity in finite time, the lifetime of the mode being T_∞ (4.5). This mode is qualitatively similar to the case $\gamma = 0$.

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Appendix A. Linear theory without the Boussinesq approximation

The linear differential equation which governs the stability of a stratified shear flow is

$$\frac{\partial}{\partial z}\left\{\rho_0 W^2 \frac{\partial \phi}{\partial z}\right\} + \rho_0 (N^2 - k^2 W^2)\phi = 0, \qquad \text{(A 1a)}$$

where

$$\frac{\partial \rho_0}{\partial z} = -\sigma \rho_0 N^2. \qquad \text{(A 1b)}$$

Here $\rho_0(z)$ is the density stratification, and σ is the Boussinesq parameter. The remaining variables have the same definitions as those given in § 2 for equations (2.1 *a*, *b*). The Boussinesq approximation is to let $\sigma \to 0$ in (A 1*a*), which then reduces to (2.1*a*). Let

$$\psi = \rho_0^{\frac{1}{2}} \phi. \qquad \text{(A 2)}$$

Then it follows that

$$\frac{\partial}{\partial z}\left\{W^2 \frac{\partial \psi}{\partial z}\right\} + \left\{N^2 - W^2\left(k^2 - \tfrac{1}{2}\sigma \frac{\partial}{\partial z}(N^2) + \tfrac{1}{4}\sigma^2 N^4\right) + \sigma N^2 W \frac{\partial W}{\partial z}\right\}\psi = 0. \qquad \text{(A 3)}$$

As in § 2, we shall suppose that, as $z \to \pm\infty$, $u_0 \to U^\pm$ and $N \to N^\pm$. Then, as $z \to \pm\infty$, it follows that

$$\psi \sim A^\pm \exp(\pm i n^\pm z) \quad \text{as} \quad z \to \pm\infty, \qquad \text{(A 4a)}$$

where

$$(n^\pm)^2 = \left(\frac{N^\pm}{W^\pm}\right)^2 - (k^2 + \tfrac{1}{4}\sigma^2 N^{\pm 4}). \qquad \text{(A 4b)}$$

The definition of $n^\pm$ reduces to (2.2*b*) as $\sigma \to 0$, and the choice of sign is again determined by the criterion $\operatorname{Im}(n^\pm) > 0$ when $\operatorname{Im} c > 0$.

In the shear layer we again suppose that u_0 and N are functions of $z^* = z/\delta$. The solution for ψ in the shear layer is found in an analogous manner to the method used for ϕ in § 2, and the result for ψ is similar to equation (2.5) for ψ. Matching proceeds as described in § 2, and the final result is the following dispersion relation

$$\hat{\mathscr{D}}(\omega, k) + \delta k^2 \hat{L} + O(\delta^2) = 0, \qquad \text{(A 5a)}$$

where

$$\begin{aligned}\hat{L} = {} & k^2 \int_0^\infty (W^2 - W^{+2})\,dz^* + k^2 \int_{-\infty}^0 (W^2 - W^{-2})\,dz^* \\ & - \int_0^\infty (N^2 + N^{+2})\,dz^* - \int_{-\infty}^0 (N^2 - N^{-2})\,dz^* \\ & + m^+ m^- W^{-2} \int_0^\infty \left(1 - \frac{W^{+2}}{W^2}\right) dz^* + m^+ m^- W^{+2} \int_{-\infty}^0 \left(1 - \frac{W^{-2}}{W^2}\right) dz^*, \end{aligned} \qquad \text{(A 5b)}$$

$$\hat{\mathscr{D}}(\omega, k) = -i m^+ k^2 W^{+2} - i m^- k^2 W^{-2}, \qquad \text{(A 5c)}$$

and

$$m^\pm = n^\pm \mp \tfrac{1}{2} i \sigma N^{\pm 2}. \qquad \text{(A 5d)}$$

Note that (A 5*a*) is formally identical with (2.8*a*) provided that $n^\pm$ in (2.8*a*) is replaced by $m^\pm$ (A 5*d*), noting that $n^\pm$ in (A 5*d*) is now given by (A 4*b*).

The vortex sheet modes are now given by

$$\hat{\mathscr{D}}(\omega_0, k) = 0. \qquad \text{(A 6)}$$

Assuming that $U^\pm = \pm 1$ and $N^\pm = 1$, it is readily verified that one solution of (A 6) is

$$\text{(i)} \quad c = 0, \quad 0 < k^2 < 1 - \tfrac{1}{4}\sigma^2. \qquad \text{(A 7)}$$

Thus mode (i) is unaffected by the Boussinesq approximation, at least in the linear theory, the only change from (1.5) being a further restriction on the wavenumber k. However, it may be shown that (A 6) has no other solutions corresponding to over-reflection modes; i.e. there are no other solutions with both c and $n^{\pm}$ real. As $\sigma \to 0$, we may perturb mode (ii), and we find that

$$\text{(ii)} \quad c = c_0\{1 - i\sigma(4k^2-1)/4k(1-2k^2) + O(\sigma^2)\}, \tag{A 8a}$$

where

$$c_0^2 = \frac{1}{2k^2} - 1. \tag{A 8b}$$

Equation (A 8a) gives the curious result that, of the two waves contained in mode (ii), the wave to the right (c_0 positive) is stabilized by the $O(\sigma)$ perturbation, while that to the left (c_0 negative) is destabilized.

Appendix B. GLM formulation of the equations of motion

In this appendix we shall give a brief outline of the generalized-Lagrangian-mean (GLM) formulation of the equations of motion due to Andrews & McIntyre (1978). For an inviscid, incompressible, stably stratified fluid the GLM formulation has been described by Grimshaw (1981), and we shall use that formulation here.

First, for any field variable ϕ, we define an Eulerian averaging operator, denoted by $\langle \ldots \rangle$; for the application described in the main body of this paper, $\langle \ldots \rangle$ denotes an average over one horizontal wavelength. Then let x, z be Lagrangian co-ordinates, and $\xi(x,z,t)$, $\zeta(x,z,t)$ be particle displacements defined so that the Eulerian co-ordinates (x', z') are related to (x, z) by the relations

$$x' = x + \xi, \quad z' = z + \zeta. \tag{B 1}$$

Then define a Lagrangian mean operator by

$$\overline{\phi}^{\mathrm{L}}(x,z,t) = \langle \phi(x+\xi, z+\zeta, t) \rangle. \tag{B 2}$$

The Lagrangian mean of ϕ is an average following the fluid motion. As shown by Andrews & McIntyre, this notion is made precise by requiring that

$$\langle \xi \rangle = \langle \zeta \rangle = 0, \tag{B 3}$$

whence it follows that (x, z) are co-ordinates which move with the Lagrangian mean velocity $(\overline{u}^{\mathrm{L}}, \overline{w}^{\mathrm{L}})$, whenever the Eulerian co-ordinates (x', z') move with the true velocity (u, w).

To obtain the equations of motion in the GLM formulation we first introduce the Jacobian of the transformation from (x, z) to (x', z'),

$$\mathscr{J} = \frac{\partial x'}{\partial x}\frac{\partial z'}{\partial z} - \frac{\partial x'}{\partial z}\frac{\partial z'}{\partial x}. \tag{B 4}$$

It may then be shown that

$$\frac{d\mathscr{J}}{dt} + \mathscr{J}\left(\frac{\partial \overline{u}^{\mathrm{L}}}{\partial x} + \frac{\partial \overline{w}^{\mathrm{L}}}{\partial z}\right) = 0, \tag{B 5a}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \overline{u}^{\mathrm{L}}\frac{\partial}{\partial x} + \overline{w}^{\mathrm{L}}\frac{\partial}{\partial z}. \tag{B 5b}$$

Here d/dt is the material derivative following the fluid motion. It follows that $\mathscr{J}$ is a Lagrangian mean quantity, and so

$$\mathscr{J} = 1+\left\langle \frac{\partial \xi}{\partial x}\frac{\partial \zeta}{\partial z}-\frac{\partial \zeta}{\partial x}\frac{\partial \xi}{\partial z}\right\rangle, \tag{B 6a}$$

and

$$\frac{\partial \xi}{\partial x}+\frac{\partial \zeta}{\partial z}+\left[\frac{\partial \xi}{\partial x}\frac{\partial \zeta}{\partial z}+\frac{\partial \zeta}{\partial x}\frac{\partial \xi}{\partial z}\right] = 0. \tag{B 6b}$$

Here, for any field quantity ϕ, we define $[\phi] = \phi-\langle\phi\rangle$ to be the Lagrangian perturbation of ϕ. The density ρ is a Lagrangian mean quantity so $\rho = \bar{\rho}^{\mathrm{L}}$ and $[\rho] = 0$, where

$$\frac{d\bar{\rho}^{\mathrm{L}}}{dt} = 0. \tag{B 7}$$

The remaining equations are

$$\bar{\rho}^{\mathrm{L}}\left(\frac{d\bar{u}^{\mathrm{L}}}{dt}+\frac{d^2\xi}{dt^2}\right)+\frac{1}{\sigma}\frac{\partial p}{\partial x'} = 0, \tag{B 8a}$$

$$\bar{\rho}^{\mathrm{L}}\left(\frac{d\bar{w}^{\mathrm{L}}}{dt}+\frac{d^2\zeta}{dt^2}\right)+\frac{1}{\sigma}\frac{\partial p}{\partial z'}+\bar{\rho}^{\mathrm{L}} = 0. \tag{B 8b}$$

A more convenient form of these equations is obtained by multiplying (B 8*a*) by $\partial x'/\partial x$ (or $\partial x'/\partial z$) and (B 8*b*) by $\partial z'/\partial x$ (or $\partial z'/\partial z$) and adding the result. Further simplification is obtained by writing

$$p = \bar{p}^{\mathrm{L}}+\sigma q+\xi\left(-\sigma\bar{\rho}^{\mathrm{L}}\frac{d\bar{u}^{\mathrm{L}}}{dt}\right)+\zeta\left(-\sigma\bar{\rho}^{\mathrm{L}}\frac{d\bar{w}^{\mathrm{L}}}{dt}-\bar{\rho}^{\mathrm{L}}\right). \tag{B 9}$$

The result of these manipulations is the following set of equations for the mean flow

$$\frac{d\bar{u}^{\mathrm{L}}}{dt}+\frac{1}{\sigma\bar{\rho}^{\mathrm{L}}}\frac{\partial \bar{p}^{\mathrm{L}}}{\partial x} = \frac{d\mathscr{P}_H}{dt}+\mathscr{P}_H\frac{\partial \bar{u}^{\mathrm{L}}}{\partial x}+\mathscr{P}_V\frac{\partial \bar{w}^{\mathrm{L}}}{\partial x}+\frac{\partial \mathscr{R}}{\partial x}, \tag{B 10a}$$

$$\frac{d\bar{w}^{\mathrm{L}}}{dt}+\frac{1}{\sigma\bar{\rho}^{\mathrm{L}}}\frac{\partial \bar{p}^{\mathrm{L}}}{\partial z}+\frac{1}{\sigma} = \frac{d\mathscr{P}_V}{dt}+\mathscr{P}_H\frac{\partial \bar{u}^{\mathrm{L}}}{\partial z}+\mathscr{P}_V\frac{\partial \bar{w}^{\mathrm{L}}}{\partial z}+\frac{\partial \mathscr{R}}{\partial z}, \tag{B 10b}$$

where

$$\mathscr{P}_H = -\left\langle\frac{d\xi}{dt}\frac{\partial \xi}{\partial x}+\frac{d\zeta}{dt}\frac{\partial \zeta}{\partial x}\right\rangle, \tag{B 10c}$$

$$\mathscr{P}_V = -\left\langle\frac{d\xi}{dt}\frac{\partial \xi}{\partial z}+\frac{d\zeta}{dt}\frac{d\zeta}{\partial z}\right\rangle, \tag{B 10d}$$

and

$$\mathscr{R} = \frac{1}{2}\left\langle\left(\frac{d\xi}{dt}\right)^2+\left(\frac{d\zeta}{dt}\right)^2\right\rangle. \tag{B 10e}$$

The equations of motion for the particle displacements are

$$\bar{\rho}^{\mathrm{L}}\frac{d^2\xi}{dt^2}+\frac{\partial q}{\partial x}-\frac{\eta}{\sigma}\frac{\partial \bar{p}^{\mathrm{L}}}{\partial x}-\xi\frac{\partial}{\partial x}\left(\bar{\rho}^{\mathrm{L}}\frac{d\bar{u}^{\mathrm{L}}}{dt}\right)-\zeta\frac{\partial}{\partial x}\left(\bar{\rho}^{\mathrm{L}}\frac{d\bar{w}^{\mathrm{L}}}{dt}\right)+\bar{\rho}^{\mathrm{L}}\left[\frac{d^2\xi}{dt^2}\frac{\partial \xi}{\partial x}+\frac{d^2\zeta}{dt^2}\frac{\partial \zeta}{\partial x}\right] = 0, \tag{B 11a}$$

$$\bar{\rho}^{\mathrm{L}}\frac{d^2\zeta}{dt^2}+\frac{\partial q}{\partial z}-\frac{\zeta}{\sigma}\frac{\partial \bar{p}^{\mathrm{L}}}{\partial z}-\xi\frac{\partial}{\partial z}\left(\bar{\rho}^{\mathrm{L}}\frac{d\bar{u}^{\mathrm{L}}}{dt}\right)-\zeta\frac{\partial}{\partial z}\left(\bar{\rho}^{\mathrm{L}}\frac{d\bar{w}^{\mathrm{L}}}{dt}\right)+\bar{\rho}^{\mathrm{L}}\left[\frac{d^2\xi}{dt^2}\frac{\partial \xi}{\partial z}+\frac{d^2\zeta}{dt^2}\frac{\partial \zeta}{\partial z}\right] = 0. \tag{B 11b}$$

The Boussinesq approximation is obtained by assuming that $\bar{\rho}^{\mathrm{L}} = \rho_0(z)+O(\sigma)$, that $\partial\rho_0/\partial z = -\sigma\rho_0 N^2$ (A 1*b*), and then taking the limit $\sigma \to 0$. In the resulting equations, $\rho_0(z)$, where it occurs explicitly, is assumed to be a constant, which we choose to be 1.

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